2D system example (Deep Reach)

Problem Setup:

1. system dynamics: $\dot{x} = v_x, \dot{y} = v_y$
2. control bounds: $\|v_x, v_y\| \leq 0.5$
3. Compute the BRT where the target set is a circle of radius 0.75 around the origin. x in [-2, 2], y in [-2, 2]
4. target set represents (a) goal states (b) unsafe states

(a) BRT in this case is defined as:
$\mathcal{R}(T) = \{x_0 : \exists u,$ s.t. x($\cdot$) satisfies $\dot{x} = f(x, u), x_0 = x(0);$
$\exists t \in [0, T],$ s.t. $x(t) \in \boldsymbol{\mathcal{L}}\}$

Target set is $\boldsymbol{\mathcal{L}} = \{(x, y) : x^2 + y^2 \leq 0.75^2\}$

Define target function $\ell(x, y)$ s.t. $\boldsymbol{\mathcal{L}} = \{(x, y) : \ell(x, y) \leq 0\}$
(usually, define $\ell(x, y)$ as a signed distance function to $\boldsymbol{\mathcal{L}}$)
$\ell(x, y) = \|x, y\| - 0.75$

The hamiltonian is $H = \min_u \{\nabla V(x(t), t) \cdot f(x, u, t)\}$
Compute H analytically, let $\boldsymbol{p} = \langle p_1, p_2 \rangle$ be the spacial gradient of the value function w.r.t. x and y. $\boldsymbol{v} = \langle v_x, v_y \rangle$. Then
$H = \min_u \{\boldsymbol{p} \cdot \boldsymbol{v}\}$ where $\boldsymbol{v}$ needs to satisfy the control bounds.
Because $\boldsymbol{p} \cdot \boldsymbol{v} = \|\boldsymbol{p}\| \|\boldsymbol{v}\| cos(\theta)$, the hamiltonian is minimized when
(1) $\boldsymbol{v}$ is in the opposite direction of $\boldsymbol{p}$
(2) $\|\boldsymbol{v}\| = 0.5$ as upper bounded by the control bounds.

Now, $\theta = -\pi \Rightarrow cos(\theta) = -1$
So, $H = -0.5\|\boldsymbol{p}\|$

(b) BRT in this case is defined as:
$\mathcal{R}(T) = \{x_0 : \forall u$, s.t. x($\cdot$) satisfies $\dot{x} = f(x,u), x_0 = x(0);$
$\exists t \in [0,T]$, s.t. $x(t) \in \mathcal{L}\}$

The target set and target function remain the same as (a).

The hamiltonian is $H = \max_u \{\nabla V(x(t),t) \cdot f(x,u,t)\}$
For similar arguments in (a), H can be computed analytically.
Now, $cos(\theta) = 1$, the corresponding $H = 0.5||\boldsymbol{p}||$

With the closed-form hamiltonians, we are able to compute BRTs for both cases using deep reach.