DEFINITION **9.1**

Given two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of a parameter θ, with variances $V(\hat{\theta}_1)$ and $V(\hat{\theta}_2)$, respectively, then the *efficiency* of $\hat{\theta}_1$ relative to $\hat{\theta}_2$, denoted $\text{eff}(\hat{\theta}_1, \hat{\theta}_2)$, is defined to be the ratio

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}.$$

THEOREM **9.1**

An unbiased estimator $\hat{\theta}_n$ for θ is a consistent estimator of θ if

$$\lim_{n\to\infty} V(\hat{\theta}_n) = 0.$$

THEOREM **9.2**

Suppose that $\hat{\theta}_n$ converges in probability to θ and that $\hat{\theta}'_n$ converges in probability to θ'.

a $\hat{\theta}_n + \hat{\theta}'_n$ converges in probability to $\theta + \theta'$.
b $\hat{\theta}_n \times \hat{\theta}'_n$ converges in probability to $\theta \times \theta'$.
c If $\theta' \neq 0$, $\hat{\theta}_n / \hat{\theta}'_n$ converges in probability to θ / θ'.
d If $g(\cdot)$ is a real-valued function that is continuous at θ, then $g(\hat{\theta}_n)$ converges in probability to $g(\theta)$.

DEFINITION **9.3**

Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a probability distribution with unknown parameter θ. Then the statistic $U = g(Y_1, Y_2, \ldots, Y_n)$ is said to be *sufficient* for θ if the conditional distribution of $Y_1, Y_2, \ldots, Y_n$, given U, does not depend on θ.

Factorization Theorem

THEOREM **9.4**

Let U be a statistic based on the random sample $Y_1, Y_2, \ldots, Y_n$. Then U is a *sufficient statistic* for the estimation of a parameter θ if and only if the likelihood $L(\theta) = L(y_1, y_2, \ldots, y_n \mid \theta)$ can be factored into two nonnegative functions,

$$L(y_1, y_2, \ldots, y_n \mid \theta) = g(u, \theta) \times h(y_1, y_2, \ldots, y_n)$$

where $g(u, \theta)$ is a function only of u and θ and $h(y_1, y_2, \ldots, y_n)$ is not a function of θ.

THEOREM **9.5**

The Rao–Blackwell Theorem Let $\hat{\theta}$ be an unbiased estimator for θ such that $V(\hat{\theta}) < \infty$. If U is a sufficient statistic for θ, define $\hat{\theta}^* = E(\hat{\theta} \mid U)$. Then, for all θ,

$$E(\hat{\theta}^*) = \theta \quad \text{and} \quad V(\hat{\theta}^*) \leq V(\hat{\theta}).$$

Find MVUE:

Likelihood function

↓

Factorization thm

↓

find U suff stat for θ

↓

find $\underbrace{f(U)}_{\text{MVUE}}$ unbiased for θ

Method of Moments

Choose as estimates those values of the parameters that are solutions of the equations $\mu'_k = m'_k$, for $k = 1, 2, \ldots, t$, where t is the number of parameters to be estimated.

$\mu'_k = \mathbb{E}(Y^k)$ $\quad$ $m'_k = \frac{1}{n}\sum_{i=1}^{n} Y_i^k$

list system of equations, solve for the params

$\mu'_1 = m'_1, \ldots, \mu'_t = m'_t$

Method of Maximum Likelihood

Suppose that the likelihood function depends on k parameters $\theta_1, \theta_2, \ldots, \theta_k$. Choose as estimates those values of the parameters that maximize the likelihood $L(y_1, y_2, \ldots, y_n \mid \theta_1, \theta_2, \ldots, \theta_k)$.

Likelihood function

↓

take derivative (log-likelihood)

set to 0

↓

solve for params