$x^2+x+1=0$, compute $x^{77}+x^{76}+x^{75}+x^{74}+x^{73}$

$$\begin{aligned} & x^{73}(x^4+x^3+x^2+x+1) \\ = & x^{73}(x^4+x^3) \\ = & x^{76}(x+1) \\ = & -x^{78} \\ = & -1 \end{aligned}$$

$$x^3+x^2+x=0$$
$$x^3+x^2+x+1=1$$
$$x^3=1$$

Cauchy-Shwarz inequality

$$\left|\sum_k a_k b_k\right| \le \sqrt{\sum_k a_k^2}\sqrt{\sum_k b_k^2}$$

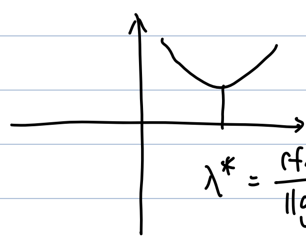
$$\left|\int f(x)g(x)\,dx\right| \le \sqrt{\int f^2\,dx}\sqrt{\int g^2\,dx} \qquad |(f,g)| \le \|f\|\,\|g\|$$

$$|\mathbb{E}\,XY| \le \sqrt{\mathbb{E}X^2}\sqrt{\mathbb{E}Y^2} \qquad (f,f) = \|f\|^2$$

$$|\mathrm{COV}(X,Y)| \le \sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}$$

Fact: $\underbrace{(f-\lambda g,\, f-\lambda g)}_{=\,\|f-\lambda g\|^2 \ge 0} = \|f\|^2 + \lambda^2\|g\|^2 - 2\lambda(f,g)$

$\lambda \in \mathbb{R}$



$2\lambda^*\|g\|^2 = 2(f,g)$

$\lambda^* = \dfrac{(f,g)}{\|g\|^2}$

$$0 \le (f-\lambda^* g,\, f-\lambda^* g) = \|f\|^2 - \frac{(f,g)^2}{\|g\|^2}$$

$$(f,g)^2 \le \|f\|^2\|g\|^2$$

Model (unknown parameter θ)

$\downarrow$

Likelihood function $L_n(\vec{X};\theta)$

distribution of the sample evaluated on n sample

$L_n(x_1,\dots,x_n;\theta)$ in pdf n pmf of $(X_1,\dots,X_n)$

$\downarrow$ (-S)

Cramer-Rao Bound ("information inequality")

regular model L_n "nice"

MLE $\longrightarrow$ asymp normal + δ-method $\rightarrow$ "formula"

$\downarrow$

invariance

$$E\hat{\theta}_n = \theta + \underbrace{b(\theta)}_{\text{bias}} = \int \hat{\theta}(\vec{x}) L_n(\vec{x};\theta)\, d\vec{x}$$

$$\frac{\partial}{\partial\theta} = 1 + b'(\theta) = \int \hat{\theta}_n(\vec{x}) \frac{\partial L_n(\vec{x};\theta)}{\partial\theta}\, d\vec{x}$$

$$= \int \hat{\theta}_n \underbrace{\frac{\partial L_n}{\partial\theta}\frac{1}{L_n}}_{= \frac{\partial \ln L_n}{\partial\theta}} L_n\, d\vec{x}$$

$$1 = \int L_n(\theta)\, d\vec{x}$$

$$\frac{\partial}{\partial\theta}: \quad 0 = \int \frac{\partial L_n}{\partial\theta}\, d\vec{x} = \int \frac{\partial L_n}{\partial\theta}\frac{1}{L_n} L_n\, dx - \mathbb{E}\frac{\partial \ln L_n}{\partial\theta}$$

by C-sh

$$\leq \sqrt{\mathrm{Var}(\hat{\theta}_n)} \sqrt{\mathbb{E}\left|\frac{\partial \ln L_n}{\partial\theta}\right|^2} \quad \frac{\partial \ln L_n}{\partial\theta}$$

$$\mathrm{Var}(\hat{\theta}_n) \geq \underbrace{\frac{(1+b'(\theta))^2}{\mathbb{E}\left|\frac{\partial \ln L_n(\vec{x};\theta)}{\partial\theta}\right|^2}}_{} = I_n(\theta) - \text{Fisher information}$$

$$MSE(\hat{\theta}_n) \geq \downarrow + (b(\hat{\theta}_n))^2$$

independent random sample

$$L_n(\vec{x};\theta) = \prod_{k=1}^{n} f(x_k;\theta) = \exp\Big(\sum_{k=1}^{n} \underbrace{\ln f(x_k;\theta)}_{= \varphi(x_k;\theta)}\Big)$$

$$\mathbb{E}\left|\frac{\partial \ln L_n}{\partial\theta}\right|^2 = \mathbb{E}\left|\sum_{k=1}^{n} \varphi_\theta(x_k;\theta)\right| = n\,\mathbb{E}\,\varphi_\theta^2(X;\theta)$$

known $\mathbb{E}\,\varphi_\theta(X_k;\theta)=0$

$$\mathbb{E}\,\varphi_\theta(x)=0=\int \varphi_\theta(x;\theta)\,f(x;\theta)\,dx$$

$$\frac{\partial}{\partial\theta}:\quad 0=\int \varphi_{\theta\theta}\,f+\frac{\varphi_\theta f_\theta}{f}\,f\,dx$$

φ_θ

$$0=\mathbb{E}\,\varphi_{\theta\theta}+\mathbb{E}\,|\varphi_\theta|^2$$

$$\Rightarrow\quad I_n(\theta)=-n\,\mathbb{E}\,\frac{\partial \ln f(x;\theta)}{\partial\theta^2}$$

Def: MLE θ_n^* is $\underset{\theta}{\operatorname{argmax}}\; L_n(\vec{x};\theta)$

regular model: $\frac{\partial L_n}{\partial\theta}(\vec{x};\hat{\theta}_n^*)=0$ $\xleftrightarrow{\text{independent random sample}}$ $\sum_{k=1}^{n}\varphi_\theta(x_k;\theta_n^*)=0$

and $\frac{\partial L_n}{\partial\theta^2}(\vec{x};\hat{\theta}_n^*)<0$

$\sum_{k=1}^{n}\varphi_\theta(x_k;\theta_n^*)=0$ $\qquad\theta_0\perp$ true param $\qquad I(\theta)=\mathbb{E}\,|\varphi_\theta(x)|^2$

$\mathbb{E}\,\varphi_\theta(X_k;\theta_n^*)=0$ $\qquad\qquad=-\mathbb{E}\,\varphi_{\theta\theta}(x)$

LLN: $\frac{1}{n}\sum_{k=1}^{n}\varphi_\theta(x_k;\theta_0)\xrightarrow{a.s.}0$ $\qquad$ LLT $-\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\varphi_\theta(x_k;\theta_0)\xrightarrow{d}N(0,I(\theta_0))$

Ex: $X\sim B(1,\theta)$

$I(\theta)-?\qquad \frac{1}{\theta(1-\theta)}$

$f(x;\theta)=\theta^x(1-\theta)^{1-x}$

$\varphi(x;\theta)=x\ln\theta+(1-x)\ln(1-\theta)$

$\varphi_\theta=\frac{x}{\theta}-\frac{1-x}{1-\theta}$

$\varphi_{\theta\theta}=-\frac{x}{\theta^2}-\frac{1-x}{(1-\theta)^2}$

$$\mathbb{E}\,\varphi_{\theta\theta}(X,\theta)=-\frac{\mathbb{E}X}{\theta^2}-\frac{1-\mathbb{E}X}{(1-\theta)^2}=-\frac{1}{\theta}-\frac{1}{1-\theta}=-\frac{1}{\theta(1-\theta)}$$