Ex. $\int_0^1 \sqrt{1-\sqrt{x}}\,dx$ can be computed using Beta function

$u=\sqrt{x}$, $x=u^2$, $dx=2u\,dx$

$2\int_0^1 u\sqrt{1-u}\,du$ $\qquad \Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(n+1)=n!$

$$= 2B(2,\tfrac{3}{2}) = \frac{2\Gamma(2)\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})} = \frac{\cancel{2}\cdot 1!\cdot \cancel{\frac{1}{2}}\cdot\cancel{\Gamma(\frac{1}{2})}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\cancel{\frac{1}{2}}\cdot\cancel{\Gamma(\frac{1}{2})}} = \frac{8}{15}$$

X – r.v. pdf $f(x)$ $\qquad$ $\mathbb{E}|X|^r$ vs $(\mathbb{E}|X|)^r$

$\mathbb{E}\,h(x) = \int_{-\infty}^{\infty} h(x)f(x)\,dx$ $\qquad$ $h(x)=|x|^r$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}|x|^r dx = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-\frac{x^2}{2}} x^{r-1}\cdot x\,dx$$

$u=\frac{x^2}{2}$, $du = x\,dx$, $x=\sqrt{2u}=(2u)^{\frac{1}{2}}$

$$\frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-u}(2u)^{\frac{r-1}{2}}du$$

$$= \frac{2\cdot 2^{\frac{r-1}{2}}}{\sqrt{2\pi}}\int_0^{\infty} u^{\underbrace{\frac{r+1}{2}}_{=t}-1} e^{-u}\,du = \frac{2\cdot 2^{\frac{r-1}{2}}}{\sqrt{2\pi}}\Gamma\left(\frac{r+1}{2}\right)$$

$\underline{r=4}$: $\frac{2\cdot 2^{\frac{r-1}{2}}}{\sqrt{2\pi}}\Gamma(\frac{5}{2}) = \frac{3}{2}\,\frac{1}{2}\,\Gamma(\frac{1}{2}) = 3$

$\Gamma(t) = \int_0^{\infty} u^{t-1}e^{-u}du$ $\qquad$ why important?

$Z \sim N(0,1)$

$\mathbb{E}Z^2 = 1$ $\quad \mathbb{E}Z^4 = 3$

$Var(Z^2) = 3-1 = 2$

$\chi_n^2 = Z_1^2 + \cdots + Z_n^2$ $\qquad Z_1,\dots,Z_n$ – i.i.d $N(0,1)$

$\mathbb{E}\, X_n^2 = n$

$\mathrm{Var}(X_n^2) = \sum_{k=1}^{n} \mathrm{Var}(Z_k^2) = 2n$

$N(0,1)$

cauchy : $\frac{N(0,1)}{N(0,1)}$ $\quad$ $X_n^2 = \sum_{i=1}^{n} (N(0,1))_i^2$

$t_n = \frac{N(0,1)}{\sqrt{\frac{X_n^2}{n}}}$ $\quad$ $F_{m,n} = \frac{(X_m^2/m)}{(X_n^2/n)}$

X, Y — i.i.d $N(0,1)$

$R^2 = x^2 + y^2 \sim \exp(\text{mean } 2) = X_2^2$

pdf $\propto e^{-x/2}$ $\quad$ exact in $\frac{1}{2} e^{-\frac{x}{2}}$

$f(x) = \frac{1}{2} e^{-\frac{x}{2}} \sim f(x) = \alpha e^{-\frac{x}{2}}$ (same)

Gamma(a, b) has pdf $f_{a,b} \propto x^{a-1} e^{-bx}$

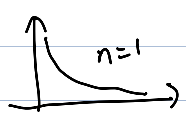
$f_{a,b} = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$

$X_n^2 = \mathrm{Gamma}(\frac{n}{2}, 2)$

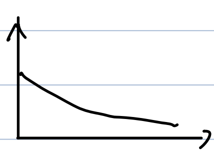
pdf of X_n^2 $\quad \propto x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$

$n = 1, 2, 3, \cdots$

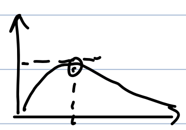
$n=1 \to \frac{1}{\sqrt{x}}$



$n=2$



$n=3 \to \sqrt{x}$



$n=4 \to x$

$n=5 \to x^{\frac{3}{2}}$

$n = 5, 6, 7$