$X \sim \Gamma(\theta)$

N-P

$H_0 : \theta = \theta_0$

$H_1 : \theta = \theta_1 > \theta_0$

$$f(x;\theta) = e^{-\theta}\frac{\theta^x}{x!} \qquad x = 0, 1, 2, \cdots$$

$$L_n(\vec{x};\theta) = e^{-\theta}\frac{\theta^{x}}{x_1!} \cdots e^{-\theta}\frac{\theta^{x}}{x_n!}$$

$$= \frac{e^{-n\theta}\,\theta^{\left(\sum_{k=1}^{n} X_k\right) = n\bar{x}_n}}{\prod_{k=1}^{n} X_k!} \quad \propto e^{-n\theta + n\bar{x}_n \ln\theta}$$

$$\varphi_n^* = \frac{L_n(\vec{x}_n;\theta_0)}{L_n(\vec{x}_n;\theta_1)} = e^{-n(\theta_0-\theta_1)+n\bar{x}_n(\ln\theta_0 - \ln\theta_1)}$$

reject H_0 if φ^* is small $\Leftrightarrow$ $\bar{x}_n$ is large

Note: $n\bar{x}_n = \sum_{k=1}^{n} X_k \overset{d}{=} g(n\theta_0)$ under H_0

To have N-P test of size α, reject H_1 if $\sum_{k=1}^{n} X_k > (\Gamma(n\theta_0))_\alpha'$

L-R Test

$$\lambda_n = \frac{L_n(\vec{x};\theta_0)}{L_n(\vec{x};\hat{\theta}_n^*)} \qquad \begin{array}{l} H_0 : \theta \le \theta_0 \\ H_a : \theta > \theta_0 \end{array}$$

MLE $= \bar{x}_n$

$$\lambda_n = \frac{\sup_{0<\theta\le\theta_0} L_n}{\sup_{\theta>0} L_n} = \begin{cases} 1 & \bar{x}_n < \theta_0 \\ & \end{cases}$$

if $\bar{x}_n < \theta$, then do not reject H_0.

$$\Rightarrow \lambda_n = \exp\left(-n(\theta_0 - \bar{x}_n) + n\bar{x}_n(\ln\theta_0 - \ln\bar{x}_n)\right)$$

$$= \exp\left(-n f(\bar{x}_n - \theta_0)\right)$$

$$= \exp\left(-n\left(-(\bar{x}_n - \theta_0) + \bar{x}_n \ln\frac{\bar{x}_n}{\theta_0}\right)\right)$$

$$f(t) = -t + (t+\theta_0)\ln\left(\frac{t+\theta_0}{\theta_0}\right)$$

$$\approx -t + (t+\theta_0)\left(\frac{t}{\theta_0} - \frac{t^2}{2\theta_0}\right) = \frac{t^2}{2\theta_0}$$

λ_n is small if $\bar{x}_n - \theta_0$ is large

$\Rightarrow$ same test as N-P

$$-2\ln\lambda_n \approx \frac{n(\bar{x}_n-\theta_0)^2}{\theta_0}$$

$$= \left(\frac{\sqrt{n}\,(\bar{x}_n-\theta_0)}{\sqrt{\theta_0}}\right)^2 = \chi^2_1$$

〃 $N(0,1)$