$2^x = 3^y = 1296$, compute $\frac{x+y}{xy} = \frac{1}{4}$

$2^x = e^{x\ln 2}$ $\quad 3^y = e^{y \ln 3}$

$1296 = 6^4 = e^{4\ln 6}$

$x = \frac{4\ln 6}{\ln 2}$ $\quad y = \frac{4 \ln 6}{\ln 3}$ $\qquad \ln 6 = \ln 2 + \ln 3$

$\frac{x+y}{xy} = \frac{1}{4}$

The best (unbiased) estimator?

1) achieve lower bound on C-R:

$$Var(\hat{\theta}_n) = \frac{1}{I_n(\theta)}$$

2) better: R-B (David Blackwell)

if U^* is a suff stat, then $\mathbb{E}(\hat{\theta}_n | U_n^*)$ is "better".

Factorization Theorem:

U_n^* is suff for $\theta \Leftrightarrow L_n(\vec{x}; \theta) = g(\vec{x})\, h(U_n^*(\vec{x}); \theta)$

Ex. $X \sim U(0, \theta)$ $\quad L_n(\vec{x}; \theta) = \mathbb{1}\{0 < x_{(n)} < \theta\}$

$\Rightarrow x_{(n)} = \max(x_1, \dots, x_n)$ is suff for θ

1) $X_{(n)}$ is complete stat
2) $2\bar{X}_n$ is the MM estimator for θ
3) $\mathbb{E}(2\bar{X}_n | X_{(n)}) = \frac{n+1}{n} X_{(n)}$ $\qquad$ UMVUE - uniform MVUE

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1) sample is a suff stat — not helpful
2) if $\bar{X}_n$ is suff $\Leftrightarrow X_1 + \dots + X_n$ is suff

then so is $(X_1+X_2, X_2+X_3, \dots)$

3) minimum suff stat: U_n^{**} mean $U_n^{**} = h(U_n^*)$ out of all suff stat

4) property of conditional expectation "small end big"

$$\mathbb{E}(\mathbb{E}(X|Y,Z)|Y) = \mathbb{E}(X|Y)$$

$\Rightarrow$ UMVUE is $\mathbb{E}(\hat{\theta}_n | U_n^{**})$

Def: Stat U_n is complete if $\mathbb{E}\, g(U_n) = 0$ for all θ, then $g \equiv 0$

Ex. $(X_1, \cdots, X_n)$ is $N(0,1)$
is not complete $\mathbb{E}(X_1 - X_2) = 0$

Theorem: for all practical purposes, complete stat $\Rightarrow$ min suff

Lehman-Scheffe Theorem:
If U_n^{**} is complete stat, then $\mathbb{E}(\hat{\theta}_n | U_n^{**})$ is UMVUE

Ex. $X_{(n)}$ is complete

$$\mathbb{E}\, g(X_{(n)}) = \frac{n}{\theta^n} \int_0^\theta t^{n-1} g(t)\,dt = 0 \quad \forall\theta \Rightarrow g \equiv 0$$

$$P(X_{(n)} < t) = (P(X < t))^n = \left(\frac{t}{\theta}\right)^n \rightarrow \frac{n t^{n-1}}{\theta^n}$$

Exponential family - of pdfs/ pmfs

$$f(x;\theta) = \exp(a(\theta) u(x) + b(\theta) + \underbrace{c(x)}_{\text{indep of } \theta}) \, I\{x \in A\}$$

if range of a contains an open set, then
$U_n^{*} = \sum_{k=1}^{n} u(X_k)$ is complete suff stat for θ

$\tilde{\theta} = a(\theta)$ is called "natural parameter"

Ex. (not) $U(0,\theta)$ $\quad f(x;\theta) = \frac{1}{\theta} I\{0 < x < \theta\}$

Ex. (Yes) $N(0,\theta)$ $\quad f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{x^2}{2\theta}\right) = \exp\left(-\frac{x^2}{2\theta} \underbrace{- \frac{1}{2}\ln(2\pi\theta)}_{b}\right)$

$$a(\theta) = \frac{1}{\theta}$$

then, $\sum_{k=1}^{n} X_k^2$ is complete suff for θ

$\Rightarrow$ UMVUE of θ is $\frac{1}{n}\sum_{k=1}^{n} X_k^2$

$\mathbb{E} X_k^2 = \theta$