$$\left(\frac{1+\sqrt{3}i}{2}\right)^{7777} \qquad i=\sqrt{-1}, \quad i^2=-1$$

$$= \left(\frac{1+\sqrt{3}i}{2}\right)^{7776+1} \qquad \frac{1+\sqrt{3}i}{2} = e^{i\frac{\pi}{3}} \quad \text{Euler}$$

$$= \frac{1+\sqrt{3}i}{2}$$

Model (θ)

$L_n(\vec{x};\theta) \to$ regular model $\xrightarrow{\text{indep rand sample}}$ $\sqrt{n}(\hat{\theta}_n^* - \theta_0) \xrightarrow{d} N\left(0, \frac{1}{I(\theta)}\right)$

pdf/pmf $f(x;\theta)$

MLE:

$\hat{\theta}_n^* = \underset{\theta}{\arg\max}\, L_n(\vec{x};\theta)$

invariance property: $\widehat{(h(\theta))}_n^* = h(\hat{\theta}_n^*)$

Fisher information

$I_n(\theta) = \mathbb{E}\left|\frac{\partial \ln L_n(\vec{x};\theta)}{\partial\theta}\right|^2 \to$ Cramer-Rao

if $E\,\hat{\theta}_n = \theta$, then $\operatorname{Var}\hat{\theta}_n \geq \frac{1}{I_n(\theta)}$

$\sqrt{I_n(\theta_0)}\,(\hat{\theta}_n^* - \theta_0) \xrightarrow{d} N(0,1)$

$\varphi(x;\theta) = \ln f(x;\theta)$

$\varphi_\theta(x;\theta) = \frac{\partial\varphi}{\partial\theta}$

$\mathbb{E}\,\varphi_\theta(x;\theta) = 0$

$I_n(\theta) = n\,I(\theta)$

$I(\theta) = -\mathbb{E}\,\varphi_{\theta\theta}(x;\theta)$

$\hat{\theta}_n^*$ satisfies

$\frac{\partial L_n}{\partial\theta} = 0$

$\sum_{k=1}^{n} \varphi_\theta(X_k;\hat{\theta}_n^*) = 0$

LLN $\quad \frac{1}{n}\sum_{k=1}^{n}\varphi_\theta(X_k;\theta_0) \xrightarrow{a.s.} 0$

CLT $\quad \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\varphi_\theta(X_k;\theta_0) \xrightarrow{d} N(0, I(\theta_0))$

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\underbrace{\left(\varphi_\theta(X_k;\theta_0) - \varphi_\theta(X_k;\hat{\theta}_n^*)\right)}_{-\varphi_{\theta\theta}(X_k;\theta_0)(\hat{\theta}_n^*-\theta_0)} \xrightarrow{d} N(0, I(\theta_0))$$

$$\frac{1}{n}\sum_{k=1}^{n}\varphi_{\theta\theta}(X_k;\theta_0) \xrightarrow{a.s.} I(\theta_0) = -\mathbb{E}\,\varphi_{\theta\theta}(X;\theta_0)$$

$$\sqrt{n}\,(\hat\theta_n^* - \theta_0) \xrightarrow{d} \frac{1}{I(\theta_0)} N(0, I(\theta_0)) = N\left(0, \frac{1}{I(\theta_0)}\right)$$

$n \to \infty$ consistency $\cancel{\longleftrightarrow}$ unbiased fixed n

$\hat\theta_n \xrightarrow[n\to\infty]{} \theta$

$\mathbb{E}\,\hat\theta_n = \theta \Rightarrow \mathbb{E}(\hat\theta_n + (X_1 - X_2)) = \theta$

$\frac{n+1}{n} X_{(n)} \to \theta$

$U(0,\theta)$

"The best estimator"

$\longrightarrow$ unbiased

option 1: lower bound in C-R hardly ever happen

Example: $X \sim B(1,p)$ $p = \theta$

$\bar X_n = \hat\theta_n$ $\quad Var(\bar X_n) = \frac{p(1-p)}{n} = \frac{1}{I_n(\theta)}$

$I_n(\theta) = \frac{n}{p(1-p)}$

option 2: improve an existing unbiased estimator

Rao-Blackwell: if $\hat\theta_n$ is unbiased and U_n is a sufficient statistic for θ, then,

$\mathbb{E}(\hat\theta_n \mid U_n)$ is a better estimator

(Recall $Var(X) = \mathbb{E}\,Var(X|Y) + Var(\mathbb{E}(X|Y))$)

Def: U_n is sufficient for θ if conditional dist. of the sample $(X_1, \dots, X_n)$ given U_n doesn't depend on θ.

Factorization theorem: U_n is sufficient for θ

$$\Leftrightarrow L_n(X_1, \dots, X_n; \theta) = g(X_1, \dots, X_n)\, h(U_n; \theta)$$

example: $\bar{X}$ is sufficient for μ is $N(\mu, 1)$

$$L_n \propto \exp\left(-\frac{1}{2}\sum_{k=1}^{n}(X_k-\mu)^2\right) = \exp\left(-\frac{1}{2}\sum_{k=1}^{n}X_k^2\right)\exp\left(\mu\sum_{k=1}^{n}X_k \frac{n\mu^2}{2}\right)$$