9.69 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function

$$f(y \mid \theta) = \begin{cases} (\theta+1)y^{\theta}, & 0 < y < 1;\ \theta > -1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find an estimator for θ by the method of moments. Show that the estimator is consistent. Is the estimator a function of the sufficient statistic $-\sum_{i=1}^{n} \ln(Y_i)$ that we can obtain from the factorization criterion? What implications does this have?

MM: sample moment = population moment

$$m_1 = \frac{1}{n}\sum_{k=1}^{n} Y_k = \mu = \mathbb{E}\,Y_1$$

$$\hat{\mu} = \bar{y}$$

$$\mu = \mathbb{E}\,Y_1 = \int_0^1 y(\theta+1)y^{\theta}\,dy = \frac{\theta+1}{\theta+2} \Rightarrow \theta = \frac{1-2\mu}{\mu-1} \Rightarrow \hat{\theta} = \frac{1-2\bar{y}}{\bar{y}-1}$$

$$\bar{y} \xrightarrow{P} \mu \Rightarrow \hat{\theta} = \frac{1-2\bar{y}}{\bar{y}-1} \xrightarrow{P} \frac{1-2\mu}{\mu-1} = \theta \quad (\text{by theorem } 9.2)$$

9.88 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function

$$f(y \mid \theta) = \begin{cases} (\theta+1)y^{\theta}, & 0 < y < 1,\ \theta > -1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the MLE for θ. Compare your answer to the method-of-moments estimator found in Exercise 9.69.

$$L(Y_1, \ldots, Y_n; \theta) = \prod_{k=1}^{n} (\theta+1)Y_k^{\theta} = (\theta+1)^n \left(\prod_{k=1}^{n} Y_k\right)^{\theta}$$

$$\ln[L(\theta)] = n\ln(\theta+1) + \theta\sum_{k=1}^{n}\ln Y_k$$

$$\frac{\partial \ln[L(\theta)]}{\partial\theta} = \frac{n}{\theta+1} + \sum_{k=1}^{n}\ln Y_k = 0$$

$$\frac{n}{\theta+1} = -\sum_{k=1}^{n}\ln Y_k$$

$$\hat{\theta} = -\frac{n}{\sum_{k=1}^{n}\ln Y_k} - 1 \qquad \frac{\partial \ln L(\theta)}{\partial\theta^2} = -\frac{n}{(\theta+1)^2} < 0$$

***9.52** Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with density function

$$f(y \mid \theta) = \begin{cases} \dfrac{3y^2}{\theta^3}, & 0 \le y \le \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ is sufficient for θ.

$$L(\theta) = \prod_{i=1}^{n} \frac{3y_i^2}{\theta^3} I\{0 < y_i < \theta\}$$

$$= \frac{3^n}{\theta^{3n}} \prod y_i^2 \, I\{y_{(n)} < \theta\}$$

$$= \underbrace{h(y_1, \cdots, y_n)}_{= \frac{1}{\theta^{3n}} \prod y_i^2} \; \underbrace{g(u, \theta)}_{= 3^n I\{y_{(n)} < \theta\}} \quad \leftarrow \text{factorization theorem}$$

$u = y_{(n)}$