8.8 Suppose that Y_1, Y_2, Y_3 denote a random sample from an exponential distribution with density function

$$f(y) = \begin{cases} \left(\frac{1}{\theta}\right) e^{-y/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Consider the following five estimators of θ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = \min(Y_1, Y_2, Y_3), \quad \hat{\theta}_5 = \overline{Y}.$$

a Which of these estimators are unbiased?

b Among the unbiased estimators, which has the smallest variance?

smallest variance

a) $\mathbb{E}\, Y_1 = \int_0^\infty y \cdot \frac{1}{\theta} e^{-y/\theta}\, dy = \cdots = \theta$

unbiased: $\hat{\theta}_1,\ \hat{\theta}_2,\ \hat{\theta}_3,\ \hat{\theta}_5$

$$\begin{aligned} F_{\hat{\theta}_4}(y) &= P(\min(Y_1, Y_2, Y_3) \le y) \\ &= 1 - P(\min(Y_1, Y_2, Y_3) > y) \\ &= 1 - P(Y_1 > y) P(Y_2 > y) P(Y_3 > y) \\ &= 1 - (1 - P(Y_1 < y))^3 \end{aligned}$$

$$\begin{aligned} f_{\hat{\theta}_4}(y) &= 3\,(1 - F_Y(y))^2 \cdot f_Y(y) \\ &= \frac{3}{\theta} e^{-3y/\theta}, \quad y > 0 \end{aligned}$$ ← exponential

$\mathbb{E}\, \hat{\theta}_4 = \frac{\theta}{3}$ ← biased

$3\hat{\theta}_4$ unbiased

8.12 The reading on a voltage meter connected to a test circuit is uniformly distributed over the interval $(\theta, \theta + 1)$, where θ is the true but unknown voltage of the circuit. Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample of such readings.

a Show that $\overline{Y}$ is a biased estimator of θ and compute the bias.

b Find a function of $\overline{Y}$ that is an unbiased estimator of θ.

c Find $\text{MSE}(\overline{Y})$ when $\overline{Y}$ is used as an estimator of θ.

a) $\mathbb{E}\,\overline{Y} = \mathbb{E}\, Y_1 = \frac{2\theta + 1}{2} \neq \theta$

$\text{Bias}(\overline{Y}) = \mathbb{E}\,\overline{Y} - \theta = \frac{1}{2}$

$\overline{Y} = \frac{1}{n} \sum_1^n Y_k$

(b) $\bar{Y} - \frac{1}{2}$

(c) $MSE(\bar{Y}) = Var(\bar{y}) + B^2(\bar{y})$

$= \frac{1}{12n} + \frac{1}{4}$

$\mathbb{E}(\bar{Y} - \theta)^2$

$= \mathbb{E}\bar{Y}^2 - 2\theta\mathbb{E}\bar{Y} + \theta^2$

$= Var(\bar{y}) + (\mathbb{E}\bar{Y})^2 - \theta(2\theta+1) + \theta^2$

$= \frac{1}{12n} + \frac{(2\theta+1)^2}{4} - \theta = \frac{1}{12}n + \frac{1}{4}$

8.18 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from a population with a uniform distribution on the interval $(0, \theta)$. Consider $Y_{(1)} = \min(Y_1, Y_2, \ldots, Y_n)$, the smallest-order statistic. Use the methods of Section 6.7 to derive $E(Y_{(1)})$. Find a multiple of $Y_{(1)}$ that is an unbiased estimator for θ.

$F_{Y(1)} = P(\min(Y_1, \ldots, Y_n) \leq y)$

$= 1 - P(\min(Y_1, \ldots, Y_n) > y)$

$= 1 - P(Y_1 > y) \cdots P(Y_n > y)$

$= 1 - (1 - P(Y_1 < y))^n$

$f_{Y(1)} = n(1 - F_{Y_1})^{n-1} \cdot f_{Y_1}$

$= n(1 - \frac{y}{\theta})^{n-1} \frac{1}{\theta}$ $\quad 0 < y < \theta$

$\mathbb{E}Y_{(1)} = \int_0^\theta n(1 - \frac{y}{\theta})^{n-1} \frac{1}{\theta}\, dy$

$= \frac{n}{\theta} \int_0^\theta (1 - \frac{y}{\theta})^{n-1}\, dy$ $\quad -\frac{\theta}{n}(1 - \frac{y}{\theta})^n$

$= \frac{n}{\theta} \; -\frac{\theta}{n}(1 - \frac{y}{\theta})^n \Big|_0^\theta$

$= -(0 - 1)$

$= 1$

$\mathbb{E}\, \theta Y_{(1)} = \theta$ unbiased