Read chapter 8.1-8.2

$\theta \leftarrow \hat{\theta}(X_1, \cdots, X_n)$, $X_1, \cdots, X_n$ – sample

$\mathbb{E}[\hat{\theta}] = \theta \Rightarrow \hat{\theta}$ is unbiased

$B(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta$ – bias

$MSE(\hat{\theta}) = \mathbb{E}[(\hat{\theta}-\theta)^2] = Var(\hat{\theta}) + (B(\hat{\theta}))^2$

8.6 Suppose that $E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta$, $V(\hat{\theta}_1) = \sigma_1^2$, and $V(\hat{\theta}_2) = \sigma_2^2$. Consider the estimator $\hat{\theta}_3 = a\hat{\theta}_1 + (1-a)\hat{\theta}_2$.

a Show that $\hat{\theta}_3$ is an unbiased estimator for θ.

b If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent, how should the constant a be chosen in order to minimize the variance of $\hat{\theta}_3$?

a) show $\mathbb{E}[\hat{\theta}_3] = \theta$

$$\mathbb{E}[a\hat{\theta}_1 + (1-a)\hat{\theta}_2] = a\,\mathbb{E}[\hat{\theta}_1] + (1-a)\,\mathbb{E}[\hat{\theta}_2]$$
$$= a\theta + (1-a)\theta$$
$$= \theta \quad \checkmark$$

b) $$Var(\hat{\theta}_3) = Var(a\hat{\theta}_1 + (1-a)\hat{\theta}_2)$$
$$= a^2 Var(\hat{\theta}_1) + (1-a)^2 Var(\hat{\theta}_2)$$
$$= a^2\sigma_1^2 + (1-a)^2\sigma_2^2 = f(a)$$

$$f'(a) = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 0$$
$$a = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

$$f''(a) = 2\sigma_1^2 + 2\sigma_2^2 > 0 \Rightarrow \text{minimum}$$