4.73 The width of bolts of fabric is normally distributed with mean 950 mm (millimeters) and standard deviation 10 mm.

a What is the probability that a randomly chosen bolt has a width of between 947 and 958 mm?

b What is the appropriate value for C such that a randomly chosen bolt has a width less than C with probability .8531?

(a) $P(947 < X < 958)$

$= P\left(\frac{947-950}{10} < \frac{X-950}{10} < \frac{958-950}{10}\right)$

$= P(-0.3 < Z < 0.8)$

$= P(Z < 0.8) - P(Z < -0.3)$

$= 0.406$

(b) $P(X < C) = 0.8531$

$P\left(\frac{X-950}{10} < \frac{C-950}{10}\right) = P\left(Z < \frac{C-950}{10}\right) = 0.8531$

$\frac{C-950}{10} = 1.05$

$C = 960.5$

7.37 Let $Y_1, Y_2, \ldots, Y_5$ be a random sample of size 5 from a normal population with mean 0 and variance 1 and let $\overline{Y} = (1/5)\sum_{i=1}^{5} Y_i$. Let Y_6 be another independent observation from the same population. What is the distribution of

a $W = \sum_{i=1}^{5} Y_i^2$? Why?

b $U = \sum_{i=1}^{5} (Y_i - \overline{Y})^2$? Why?

c $\sum_{i=1}^{5} (Y_i - \overline{Y})^2 + Y_6^2$? Why?

(a) $W \sim \chi_5^2$

(b) $U \sim \chi_4^2$ (Th. 7.3 in textbook)

(c) $\sim \chi_5^2$

7.38 Suppose that $Y_1, Y_2, \ldots, Y_5, Y_6, \overline{Y}, W$, and U are as defined in Exercise 7.37. What is the distribution of

a $\sqrt{5}Y_6/\sqrt{W}$? Why?
b $2Y_6/\sqrt{U}$? Why?
c $2\left(5\overline{Y}^2 + Y_6^2\right)/U$? Why?

(a) $\frac{Z}{\sqrt{\chi_d^2/d}} \sim t_d$ $\qquad \frac{\sqrt{5}\,Y_6}{\sqrt{W}} = \frac{Y_6}{\sqrt{\frac{W}{5}}} \sim t_5$

(b) $\frac{2Y_6}{\sqrt{U}} = \frac{Y_6}{\sqrt{U/4}} \sim t_4$

(c) $\frac{\chi_d^2/d}{\chi_n^2/n} \sim F_{d,n}$ $\qquad \frac{2(5\overline{Y}^2+Y_6^2)}{U} = \frac{\frac{5\overline{Y}^2+Y_6^2}{2}}{\frac{U}{4}} = F_{2,4}$

$\overline{Y} \sim N(0, \frac{1}{5})$ $\qquad \overline{Y} = \frac{1}{5}\sum_{1}^{5} Y_i$

$\sqrt{5}\,\overline{Y} \sim N(0,1)$ $\qquad Var(\overline{Y}) = \frac{1}{5}$

$(\sqrt{5}\,\overline{Y})^2 \sim \chi_1^2$

7. Let X and Y be independent standard normal random variables. Define $U = X + Y$ and $V = X - Y$.
(a) Confirm that U and V are independent.
(b) Compute $E(X + 2Y|U)$ and $E(X + 2Y|V)$.

(a) $Cov(U,V) = Cov(X, X-Y) + Cov(Y, X-Y)$
$= Cov(X,X) - Cov(X,Y) + Cov(Y,X) - Cov(Y,Y)$
$= 0$

normal $\Rightarrow$ ($COV=0 \Rightarrow$ independent)

(b) $\mathbb{E}[U|U] = U$,
U, V independent $\mathbb{E}[U|V] = \mathbb{E}\,U$

$\mathbb{E}[X+2Y \mid U]$

$= \mathbb{E}[\frac{3}{2}U - \frac{1}{2}V \mid U]$

$= \mathbb{E}[\frac{3}{2}U \mid U] + \mathbb{E}[-\frac{1}{2}V \mid U]$

$= \frac{3}{2}U - \frac{1}{2}\mathbb{E}V$

$= \frac{3}{2}U$

$X = \frac{U+V}{2}$

$Y = \frac{U-V}{2}$