Example: $Y_1 = \theta + \varepsilon_1$

$Y_2 = 2\theta - \varphi + \varepsilon_2$ $\qquad \mathbb{E}\,\varepsilon_i = 0 \quad \forall i$

$Y_3 = \theta + 2\varphi + \varepsilon_3$

Find the least square estimate of θ and φ

$$X = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \qquad \beta = \begin{pmatrix} \theta \\ \varphi \end{pmatrix}$$

$$y = X\beta + \varepsilon$$

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

$$(X^T X)^{-1} = \left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}^{-1} = \frac{1}{30} \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\hat{\beta} = \frac{1}{30} \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} 5Y_1 + 10Y_2 + 5Y_3 \\ -6Y_2 + 12Y_3 \end{bmatrix}$$

$$\hat{\theta} = \frac{1}{6}(Y_1 + 2Y_2 + Y_3)$$

$$\hat{\varphi} = \frac{1}{5}(-Y_2 + 2Y_3)$$

$$SSE = (Y_1 - \theta)^2 + (Y_2 - 2\theta + \varphi)^2 + (Y_3 - \theta - 2\varphi)^2$$

12.1 $\quad \sigma_1^2 = 9 \qquad \sigma_2^2 = 25 \qquad n = 90$

find n_1, n_2 that gives max info of $\mu_1 - \mu_2$

minimize $Var(\bar{X} - \bar{Y}) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{90 - n_1}$

$$n_1 = \frac{\sigma_1}{\sigma_1 + \sigma_2} \qquad n_2 = \frac{\sigma_2}{\sigma_1 + \sigma_2}$$