$Y = X\beta + \varepsilon$

$X^T Y = X^T X \beta$

$SSE = RSE = \Sigma(y_i - \beta^T x_i)^2 = (X\beta - Y)^T (X\beta - Y)$

$\nabla_\beta SSE = X^T X \beta - X^T Y = 0$

9.23 $\quad Y_1, \cdots, Y_n \sim f(y|\theta) = \begin{cases} \frac{1}{2\theta+1} & 0 \le y \le 2\theta+1 \\ 0 & \text{else} \end{cases}$

a) find MLE of θ

$$L(Y_1, \cdots, Y_n | \theta) = \prod_{i=1}^{n} \frac{1}{2\theta+1} I(0 \le Y_i \le 2\theta+1)$$

$$= \left(\frac{1}{2\theta+1}\right)^n I\{Y_{(n)} \le 2\theta+1\}$$

- smallest θ
- $\theta \ge \frac{Y_{(n)}-1}{2} \quad \Rightarrow \quad \hat{\theta} = \frac{Y_{(n)}-1}{2}$ is MLE of θ

b) find MLE of the variance

$$\text{Var}(Y) = \frac{(2\theta+1)^2}{12}$$

$$\hat{\text{Var}}(Y) = \frac{Y_{(n)}^2}{12}$$

P4 (Sp'22)

$X_1, \cdots, X_n \sim \text{Gamma}(3, \theta)$

$$f(x|\theta) = \frac{1}{2\theta^3} x^2 e^{-\frac{x}{\theta}}, \quad x > 0$$

Construct the most powerful test, $\alpha = 0.05$

$H_0 : \theta = 1, \quad H_a : \theta = 2$

$$L_n(\vec{X};\theta) = \prod_{i=1}^{n} \frac{1}{2\theta^3} x_i^2 e^{-\frac{x_i}{\theta}}$$

$$= \left(\frac{1}{2\theta^3}\right)^n (x_1\cdots x_n)^2 \exp\left(-\frac{1}{\theta}\sum_{i=1}^{n} x_i\right)$$

$$\frac{L_n(\vec{x};1)}{L_n(\vec{x};2)} = \frac{\left(\frac{1}{2}\right)^n (x_1\cdots x_n)^2 \exp\left(-\sum_{i=1}^{n} x_i\right)}{\left(\frac{1}{16}\right)^n (x_1\cdots x_n)^2 \exp\left(-\frac{1}{2}\sum_{i=1}^{n} x_i\right)}$$

$$= 8^n \exp\left(-\frac{1}{2}\sum_{i=1}^{n} x_i\right) < k$$

reject H_0 when $\sum_{i=1}^{n} x_i > k'$ $\qquad n\bar{x} > k'$

$0.05 = \alpha = p(\text{reject } H_0 \text{ when } H_0 \text{ is true})$

$= p(n\bar{X}_n > k')$

under H_0, $X_i \sim \text{Gamma}(3,1)$ $\qquad n\bar{X}_n = X_1 + \cdots + X_n \sim \text{Gamma}(3n,1)$

determine k'

MPT is to reject H_0 if $n\bar{X}_n > \text{Gamma}(3n,1)\big|_{\alpha=0.05}$

Problem 5. For the first-year students at a certain university, the correlation between SAT scores and the first-year GPA was 0.68. Assume that the distribution of the scores is jointly normal. Predict the percentile rank on the first-year GPA for a student whose percentile rank on the SAT was 40%.

$(Y|X=\ )$ $\quad y = \mu_y + \frac{\sigma_y}{\sigma_x}\rho_{xy}(x-\mu_x)$

$\frac{y-\mu_y}{\sigma_y} = \rho\frac{x-\mu_x}{\sigma_x}$ $\qquad \frac{x-\mu_x}{\sigma_x} = Z$ $\qquad P(Z \le x_0) = 0.4$

$x = -0.25$

$y_0 = \rho x_0 = (0.68)(-0.25) = -0.17$ $\qquad p(Z \le -0.17) = 0.43$