NP-Lemma : $\frac{L(\theta_0)}{L(\theta_a)} < k$

$H_0: \theta = \theta_0$ vs $H_a: \theta = \theta_a$

10.94 Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a normal distribution with *known* mean μ and unknown variance σ^2. Find the most powerful α-level test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 = \sigma_1^2$, where $\sigma_1^2 > \sigma_0^2$. Show that this test is equivalent to a χ^2 test. Is the test uniformly most powerful for $H_a : \sigma^2 > \sigma_0^2$?

$$L(\sigma^2) = \prod_{i=1}^{n} f(y_i | \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left(\frac{-\sum (y_i - \mu)^2}{2\sigma^2}\right)$$

$$\frac{L(\sigma_1^2)}{L(\sigma_0^2)} < k$$

$$\Leftrightarrow \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left(-\sum (y_i-\mu)^2 \left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)\right) < k$$

$$\exp\left(-\sum (y_i-\mu)^2 \left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2}\right)\right) < k\left(\frac{\sigma_0}{\sigma_1}\right)^n$$

$$-\sum (y_i-\mu)^2 \left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2}\right) < \ln k + n\ln\frac{\sigma_0}{\sigma_1}$$

$$\frac{(y_i-\mu)^2}{\sigma_0^2} > \left(\ln k + n\ln\frac{\sigma_0}{\sigma_1}\right)\frac{2\sigma_1^2}{\sigma_0^2 - \sigma_1^2} = k^*$$

$$\varphi = \frac{\sum (y_i-\mu)^2}{\sigma_0^2} = \sum \left(\frac{(y_i-\mu)}{\sigma_0}\right)^2 \sim \chi_n^2$$

$$\begin{aligned}\alpha &= P(\text{reject } H_0 \mid H_0 \text{ true}) \\ &= P(\varphi > k^*) \\ &= P(\chi_n^2 > \chi_{n,\alpha}^2)\end{aligned}$$

RR: $\{\varphi > \chi_{n,\alpha}^2\}$ $\Rightarrow$ UMP test for H_a

$\chi^2 > \chi_\alpha^2$, $(n-1)$ df

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{\sum (y_i - \bar{y})^2}{\sigma_0^2}$$

10.105 Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a normal distribution with mean μ (unknown) and variance σ^2. For testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_a : \sigma^2 > \sigma_0^2$, show that the likelihood ratio test is equivalent to the χ^2 test given in Section 10.9.

$$\lambda \le k \quad , \lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}$$

$$\Omega_0 = \{(\mu, \sigma^2) : \mu \in \mathbb{R},\ \sigma^2 = \sigma_0^2\}$$
$$\Omega_a = \{(\mu, \sigma^2) : \mu \in \mathbb{R},\ \sigma^2 > \sigma_0^2\}$$
$$\Omega = \Omega_0 \cup \Omega_a = \{(\mu, \sigma^2) : \mu \in \mathbb{R},\ \sigma^2 \ge \sigma_0^2\}$$

$$L(\Omega_0) = \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n \exp\left(-\frac{\sum (y_i - \mu)^2}{2\sigma_0^2}\right)$$

$\vdots$

$$\hat{\mu} = \bar{y}$$

$$L(\Omega) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left(\frac{-\sum (y_i - \mu)^2}{2\sigma^2}\right)$$

$\vdots$

$$\hat{\mu} = \bar{y}\ ,\ \hat{\sigma} = \frac{1}{n}\sum (y_i - \bar{y})^2$$

$$\hat{\sigma} = \max\left\{\frac{1}{n}\sum (y_i - \bar{y})^2, \sigma_0\right\}$$

$$\frac{L(\Omega_0)}{L(\Omega)} = \left(\frac{\hat{\sigma}}{\sigma_0}\right)^n \exp\left(-\frac{\sum (y_i - \bar{y})^2}{2\sigma_0^2} + \frac{\sum (y_i - \bar{y})^2}{2\hat{\sigma}^2}\right)$$

1) if $\hat{\sigma} = \sigma_0$, $\frac{L(\Omega_0)}{L(\Omega)} = 1$

2) $\hat{\sigma} = \frac{1}{n}\sum (y_i - \bar{y})^2$, $\left(\frac{\sum (y_i - \bar{y})^2}{n\sigma_0^2}\right)^{n/2} \exp\left(-\frac{\sum (y_i - \bar{y})^2}{2\sigma_0^2} + \frac{n}{2}\right) \le k$