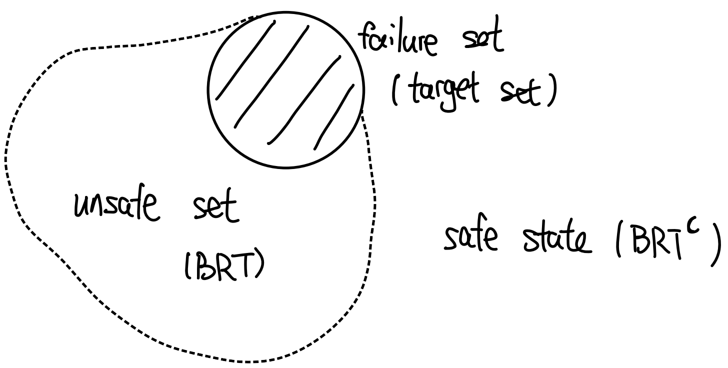
# Reachability Problem

## Backward Reachable Tube (BRT)

- The set of all starting states of the system that will eventually reach some target set despite the best control effort.

  If the target set is the failure set, then the BRT is the unsafe set.

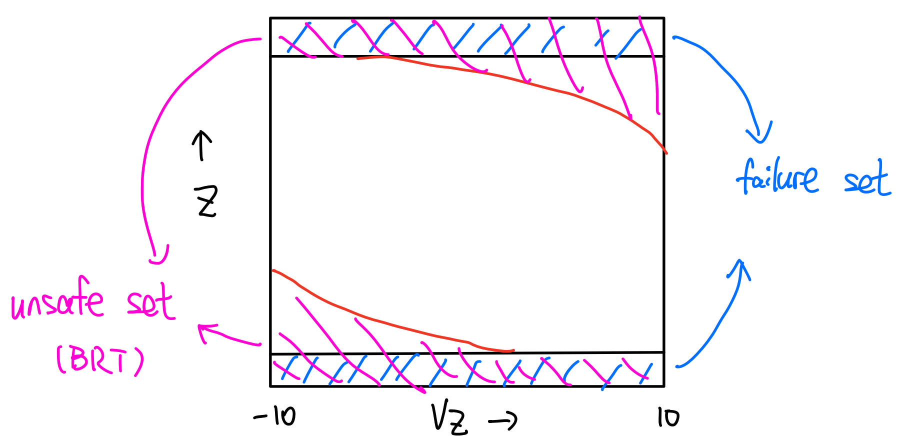



Mathematical definition:

$\mathcal{L} \subseteq \mathbb{R}^n$ be the target set (usually failure set)

$BRT(t) \subseteq \mathbb{R}^n$ be the BRT at time $t$

$BRT(t) = \{ x : \forall u(\cdot) \in \mathbb{U}_t^T, \exists d(\cdot) \in \mathbb{D}_t^T, \text{ s.t. } \xi_{x,t}^{u,d}(s) \in \mathcal{L} \text{ for some } s \in [t, T] \}$



# Hamilton-Jacobi-Reachability

- one of the (many) tools to solve reachability (compute BRT)
- Converts reachability problem into an optimal control problem

+ can handle any nonlinear dynamics
+ can handle non-deterministic uncertainty in the system
+ handle control bounds
+ handle state constraints

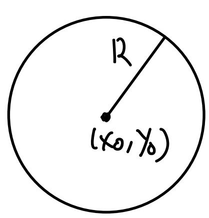
Convert BRT computation to an optimal control problem

Level set approach.

(1) define a target function $\ell(x)$ that implicitly represents the target set $\mathcal{L} \in \mathbb{R}^n$

$$\mathcal{L} = \{x : \ell(x) \leq 0\}$$

signed distance function (popular choice of $\ell(x)$)




$$\ell(x) = \sqrt{(P_x - x_0)^2 + (P_y - y_0)^2} - R$$

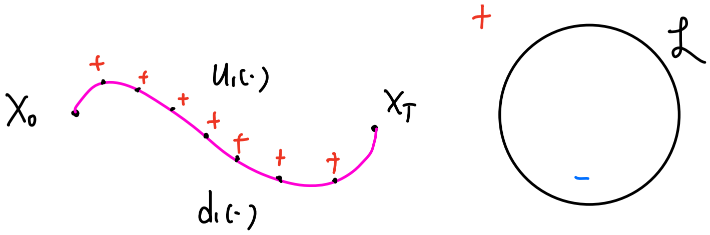
$\ell(x) > 0$ outside $\mathcal{L}$

$\ell(x) = 0$ at boundary

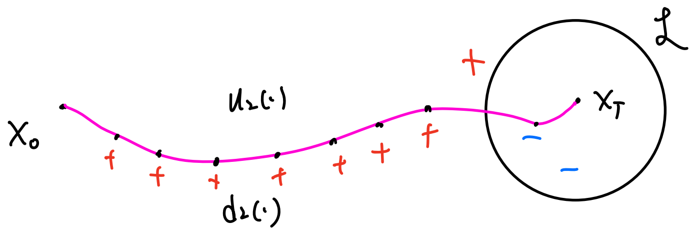
$\ell(x) < 0$ inside $\mathcal{L}$

(2) Cost function for the reachability problem

$$J(x, u(\cdot), d(\cdot), t) = \min_{s \in [t, T]} \ell(x(s))$$




$J(x, u_1(\cdot), d_1(\cdot), t) > 0$




$J(x, u_2(\cdot), d_2(\cdot), t) < 0$

$$\max_{u(\cdot)} \min_{d(\cdot)} J(x, u(\cdot), d(\cdot), t) = V(x,t)$$

$V(x,t) < 0 \Rightarrow x \in BRT(t)$

$V(x,t) > 0 \Rightarrow x \;\; BRT^c(t)$

$V(x,t) = 0 \Rightarrow x \in BRT(t)$

$$BRT(t) = \{x : V(x,t) \leq 0\}$$

$\downarrow$

BRT is the subzero level set of $V(x,t)$