- ☐ Optimal Control Problem
- ☐ Calculus of Variations
- ☐ Model Predictive Control (MPC)

Optimal Control Problem

discrete-time

$$u^*_{t:T} = \arg\min_{u_{t:T}} J(x_t, u_{t:T}, t)$$

optimal control

subject to

$$x_{t+1} = f_D(x_t, u_t)$$

$$\underline{u} \le u_t \le \bar{u}$$

continuous-time

$$u^*(\cdot) = \arg\min_{u(\cdot)} J(x(t), u(t), t)$$

subject to $\dot{x}(t) = f_c(x(t), u(t))$

$$\underline{u} \le u(t) \le \bar{u}$$

$$J(x(t), u(t), t) = \int_{\tau=t}^{T} u^2(\tau)\, d\tau$$

Calculus of Variation (CoV)

CoV deals with minimizing (maximizing) functionals.

real valued functions whose inputs are functions

cont-time OC problem falls under this category.

Step 1 : Convert the OC problem into an unconstrained problem using Lagrange multipliers.

$$\tilde{J}(x(t), u(\cdot), t, p(\cdot)) = J(x(t), u(\cdot), t) + \int_{s=t}^{T} (\dot{x}(s) - f_c(x(s), u(s)))\, \underbrace{\underbrace{p(s)}_{\text{Lagrange multiplier}}}_{\text{co-state}}\, ds$$

$$\min_{u(\cdot)} \max_{p(\cdot)} \tilde{J}(x(t), u(\cdot), t, p(\cdot))$$

step 2: solve the unconstrained OC problem using the first-order optimality conditions

$$\frac{\partial \tilde{J}}{\partial u} = 0, \quad \frac{\partial \tilde{J}}{\partial p} = 0$$

step 3: These first-order conditions can be used to obtain a "local optimal" solution.

$$u^*(\cdot), \; p^*(\cdot)$$

Pros	Cons
- a reasonable solution using tools from optimization - one of the few methods that applies to continuous time problems	- have to solve ODE, which could be hard - only "locally optimal" solution unless the OC problem is convex

Model Predictive Control

- mainly for discrete-time OC

$$\min_{u_{t:T-1}} J(x_t, u_{t:T-1}, t) \quad \text{subject to} \quad \begin{array}{ll} x_{s+1} = f_D(x_s, u_s) & \forall s \in \{t, \dots, T\} \\ \underline{u} \le u_s \le \bar{u} & x_t \text{ is given} \end{array}$$

- $\{u_t, \dots, u_{T-1}\} \rightarrow$ explicit

  $\{x_t, \dots, x_T\} \rightarrow$ implicit

  \# of variables : $(T-1-t) \times (n_x, n_u)$

  size of x size of u

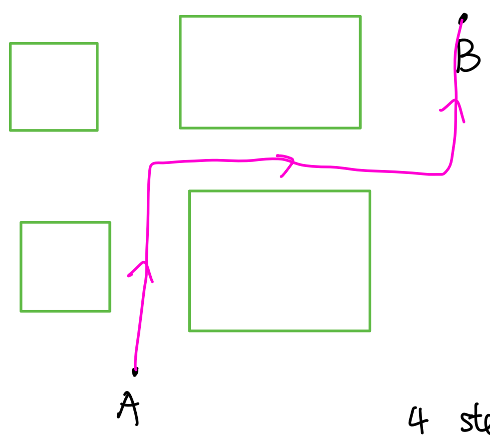
- MPC typically gives "local optimal" solution

Feedback control using MPC

$\{u_t^*, \dots, u_{T-1}^*\}$

$\downarrow$ apply on the real system

$x_{t+1} \rightarrow$ solve MPC again



$$\min_{u_{t:T}} \sum_{s=t}^{T} (p_x^s - g_x^s)^2 + (p_y^s - g_y^s)^2$$

desired trajectory from A to B

subject to human dynamics

4 steps $\uparrow$

5 steps $\rightarrow$

3 steps $\uparrow$

step input $\rightarrow$ $\boxed{\text{human body}}$ $\rightarrow$ next position

We solve a slightly different OC problem at each time step.

Time-step   optimization problem

@ time t

optimization horizon = $T-t-1$

$$u^*_{t:T} = \underset{u_{t:T-1}}{\arg\min}\ J(x_t, u_{t:T-1}, t)$$

s.t. dynamics and control bounds

$\downarrow$

apply u^*_t to the real system and obtain x_{t+1}

@ time $t+1$

optimization horizon = $T-t-2$

$$u^*_{t+1:T} = \underset{u_{t+1:T-1}}{\arg\min}\ J(x_{t+1}, u_{t+1:T-1}, t+1)$$

s.t. dynamics and control bounds

$\downarrow$

apply u^*_{t+1} to the real system and obtain x_{t+2}

This gives a feedback control.

Receding-horizon control