Discrete - time DP

$$V(x,T) = \max_u \{ \mathcal{L}(x,u) + V(x_+, t+1) \}$$

$x_+ = f(x,u)$

$$u^*(x_0, t^*) = \arg\max_u \{ \mathcal{L}(x_0^*, u) + V(x_+, t^*+1) \}$$

Continuous - time DP

$$\frac{\partial V}{\partial t} + \max_u \{ \mathcal{L}(x,u) + \frac{\partial V}{\partial x} \cdot f(x,u) \} = 0$$

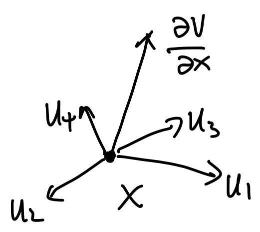
$$u^*(x,t) = \arg\max_u \{ \mathcal{L}(x,u) + \frac{\partial V}{\partial t} \cdot f(x,u) \}$$

$$\min \{ \ell(x) - V(x,t), \frac{\partial V}{\partial t} + \max_u \frac{\partial V}{\partial x} \cdot f(x,u) \} = 0$$

$$V(x,T) = \ell(x)$$

$$\underline{u^*(x,t)} = \arg\max_u \frac{\partial V}{\partial x} \cdot f(x,u)$$

safety controller



intuitively, u^* is trying to move the system towards an increasing value. i.e. steers away from the failure set

* if the system starts in BRT^c,

then u^* will keep the system outside the failure set.

If BRT converges, $V(x,\tau) \equiv V^*(x) \quad \forall \tau < t$

then $$\boxed{u^*(x) = \arg\max_u \frac{\partial V^*(x)}{\partial x} \cdot f(x,u)}$$

For control affine system: $f(x,u) = f_1(x) + f_2(x)u$

$$\frac{\partial V^*(x)}{\partial x} \cdot f(x,u) = \frac{\partial V^*(x)}{\partial x} \cdot f_1(x) + \frac{\partial V^*(x)}{\partial x} f_2(x) u$$

$$u^*(x) = \arg\max_u \left\{ \frac{\partial V^*}{\partial x} \cdot f_1(x) + \frac{\partial V^*}{\partial x} \cdot f_2(x) u \right\}$$

$$= \arg\max_u \frac{\partial V^*}{\partial x} \cdot f_2(x) u$$

$X = \begin{bmatrix} z \\ v \end{bmatrix} \qquad \dot{z} = v \qquad |u| \leq \bar{u}$

$\dot{v} = ku + g$

$$f(x,u) = \begin{bmatrix} v \\ ku+g \end{bmatrix} = \underbrace{\begin{bmatrix} v \\ g \end{bmatrix}}_{f_1(x)} + \underbrace{\begin{bmatrix} 0 \\ k \end{bmatrix}}_{f_2(x)} u$$

$$u^*(x) = \arg\max_u \frac{\partial V^*}{\partial x} \cdot f(x,u) \qquad \frac{\partial V^*}{\partial x} = \begin{bmatrix} P_1(x) \\ P_2(x) \end{bmatrix}$$

$$= \arg\max_u \begin{bmatrix} P_1(x) & P_2(x) \end{bmatrix} \left(\begin{bmatrix} v \\ g \end{bmatrix} + \begin{bmatrix} 0 \\ k \end{bmatrix} u \right)$$

$$= \arg\max_u \left[(P_1(x) v + P_2(x) g) + P_2(x) k u \right]$$

$$= \arg\max_u P_2(x) k u$$

$$u^*(x) = \begin{cases} \bar{u} & \text{if } P_2(x) \geq 0 \\ -\bar{u} & \text{if } P_2(x) < 0 \end{cases} \qquad \leftarrow \text{Bang-bang control}$$

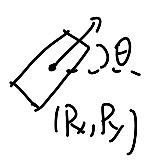
$$V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

$$\frac{\partial V}{\partial x} : \mathbb{R}^n \qquad\qquad V : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$$

$$\frac{\partial V}{\partial t} : \mathbb{R} \qquad\qquad \frac{\partial V}{\partial x} = \begin{bmatrix} \frac{\partial V}{\partial z} \\ \frac{\partial V}{\partial v} \end{bmatrix} = \begin{bmatrix} P_1(x) \\ P_2(x) \end{bmatrix}$$

Ex planar car

$$X = \begin{bmatrix} P_x \\ P_y \\ \theta \end{bmatrix} \qquad f(x,u) = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ w \end{bmatrix}$$



control : $u \triangleq w$ s.t. $|w| \leq \bar{w}$

$$f(x,u) = \underbrace{\begin{bmatrix} v\cos\theta \\ v\sin\theta \\ 0 \end{bmatrix}}_{f_1(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{f_2(x)} w$$

$$\frac{\partial V^*}{\partial x} = \begin{bmatrix} P_1(x) \\ P_2(x) \\ P_3(x) \end{bmatrix}$$

$$u^*(x) = \arg\max_u \frac{\partial V^*}{\partial x} \cdot f(x,u)$$

$$= \arg\max_u \frac{\partial V^*}{\partial x} \cdot f_2(x) u$$

$$= \arg\max_u P_3(x)\, u$$

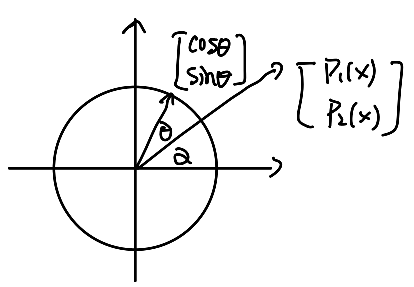
$$= \begin{cases} \bar{w} & \text{if } P_3(x) \geq 0 \\ -\bar{w} & \text{if } P_3(x) < 0 \end{cases}$$

Ex $\quad X = \begin{bmatrix} P_x \\ P_y \end{bmatrix} \quad f(x,u) = \begin{bmatrix} v\cos\theta \\ v\sin\theta \end{bmatrix} \quad u = \theta$

$$u^*(x) = \arg\max_u \frac{\partial V^*}{\partial x} \cdot f(x,u)$$

$$= \arg\max_\theta P_1(x)\, v\cos\theta + P_2(x)\, v\sin\theta$$

$$= \arg\max_\theta v\,(P_1(x)\sin\theta + P_2(x)\cos\theta)$$



$$\theta = \alpha = \tan^{-1}\left(\frac{P_2(x)}{P_1(x)}\right)$$

Safety filtering

$u_{nom}(x) \longrightarrow$ performance / nominal controller

(A) $$u^*(x) = \begin{cases} u_{nom}(x) & \text{if system is safe} \iff V^*(x) > 0 \quad > \varepsilon \\ u^*_{safe}(x) & \text{if system is at risk} \iff V^*(x) \to 0 \quad \leq \varepsilon \end{cases}$$

in practice

(A) is least restrict safety filtering