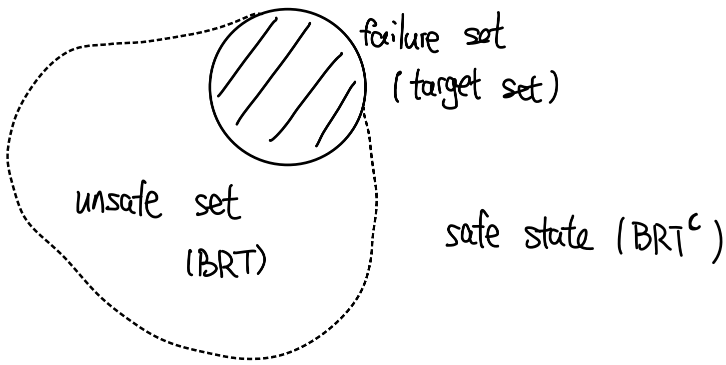
Review:



$$BRT(t) = \{ x : \forall u(\cdot) \in \mathbb{U}_t^T,\ \exists d(\cdot) \in \mathbb{D}_t^T,\ s.t.\ \xi_{x,t}^{u,d}(s) \in \mathcal{L} \text{ for some } s \in [t,T] \}$$

Hamilton - Jacobi - Reachability

$$J(x, t, u(\cdot), d(\cdot)) = \min_{s \in [t,T]} \ell(x(s))$$

$$V(x,t) = \max_{u(\cdot) \in \Gamma_u} \min_{d(\cdot) \in \Gamma_d} J(x, t, u(\cdot), d(\cdot))$$

$$BRT(t) = \{ x : V(x,t) \le 0 \} \qquad \boxed{x \in \mathcal{L} \Leftrightarrow \ell(x) \le 0}$$

$$V(x,t) = \max_{u(\cdot)} \min_{d(\cdot)} \min \Big\{ \min_{s \in [t, t+\delta]} \ell(x(s)),\ \underbrace{\min_{\tau \in [t+\delta, T]} \ell(x(\tau))}_{= J(x(t+\delta),\, t+\delta,\, u(\cdot),\, d(\cdot))} \Big\}$$

$$= \max_{u(\cdot)} \min_{d(\cdot)} \min \Big\{ \min_{s \in [t, t+\delta]} \ell(x(s)),\ V(x(t+\delta), t+\delta) \Big\}$$

Now, if δ is small

$$\min_{s\in[t,t+\delta]} \ell(x(s)) = \ell(x(t))$$

$$V(x(t+\delta), t+\delta) = V(x,t) + \frac{\partial V}{\partial t}\delta + \frac{\partial V}{\partial x}\left(x(t+\delta) - x(t)\right) + h.o.t.$$

$$= V(x,t) + \frac{\partial V}{\partial t}\delta + \frac{\partial V}{\partial x} f(x,u,d)\delta + h.o.t.$$

$$V(x,t) = \max_{u(\cdot)} \min_{d(\cdot)} \min \left\{ \ell(x(t)),\ V(x,t) + \frac{\partial V}{\partial t}\delta + \frac{\partial V}{\partial x} f(x,u,d)\delta \right\}$$

$$= \min \left\{ \max_{u(\cdot)} \min_{d(\cdot)} \ell(x(t)),\ \max_{u(\cdot)} \min_{d(\cdot)} V(x,t) + \frac{\partial V}{\partial t}\delta + \frac{\partial V}{\partial x} f(x,u,d)\delta \right\}$$

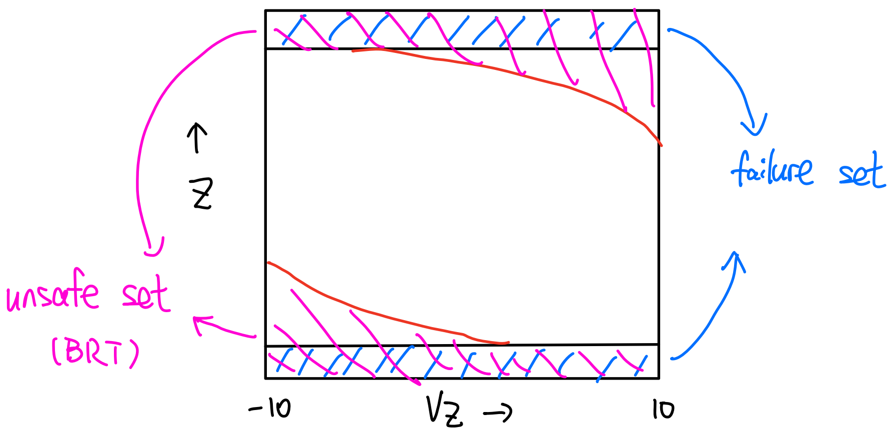
$$= \min \left\{ \ell(x(t)),\ V(x,t) + \frac{\partial V}{\partial t}\delta + \max_{u(\cdot)} \min_{d(\cdot)} \frac{\partial V}{\partial x} f(x,u,d)\delta \right\}$$

$$0 = \min \left\{ \ell(x(t)) - V(x,t),\ \left(\frac{\partial V}{\partial t} + \max_{u(\cdot)} \min_{d(\cdot)} \frac{\partial V}{\partial x} f(x,u,d) \right)\delta \right\}$$

since this holds $\forall \delta > 0$,

$$\boxed{\begin{gathered} \Rightarrow \min \left\{ \ell(x(t)) - V(x,t),\ \frac{\partial V}{\partial t} + \max_{u(\cdot)} \min_{d(\cdot)} \frac{\partial V}{\partial x} f(x,u,d) \right\} = 0 \\ V(x,T) = \ell(x) \end{gathered}}$$

HJI Variational Inequality



$$\dot{P}_z = V_z$$

$$\dot{V}_z = k_1 u + g + d$$